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ALGORITHM FOR FINDING BOUNDARY LINK SEIFERT MATRICES

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ABSTRACT

We explain an algorithm for finding a boundary link Seifert matrix for a given multivariable Alexander polynomial. The algorithm depends on several choices and therefore makes it possible to find non-equivalent Seifert matrices for a given Alexander polynomial.

Keywords: Seifert matrix; boundary link.

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1. Introduction

1.1. *Algebraic statement*

We call $A = (A_{ij})_{i,j=1,\dots,m}$ a (boundary link) Seifert matrix if A is a matrix with entries A_{ij} which are $(n_i \times n_j)$-matrices over $\mathbb{Z}$ such that $A_{ij} = A_{ji}^t$ for $i \neq j$ and $\det(A_{ii} - A_{ii}^t) = 1$ (for more details, cf. [10, 14]). Note that the n_i are necessarily even numbers. Set

$$T := \operatorname{diag}\Bigg(\underbrace{t_1,\dots,t_1}_{n_1},\dots,\underbrace{t_m,\dots,t_m}_{n_m}\Bigg)$$

then define the Alexander polynomial of A to be

$$\Delta(A) := \det(T)^{-\frac{1}{2}} \det(TA - A^t) \in \Lambda_m := \mathbb{Z}\big[t_1^{\pm 1},\dots,t_m^{\pm 1}\big].$$

This polynomial has the following well-known properties which can easily be verified from the definitions.

$$\Delta(A)(1,\dots,1) = 1,$$
$$\Delta(A)(t_1,\dots,t_m) = \Delta(A)\big(t_1^{-1},\dots,t_m^{-1}\big).$$

Now assume that Δ is a polynomial with the above properties. The goal of this paper is to give an algorithm for finding a Seifert matrix A in terms of the coefficients of Δ such that $\Delta(A) = \Delta$. In the case $m = 1$, i.e. the case of Seifert matrices for knots, an algorithm has been found by Seifert (cf. [1, 19]).

1.2. *Topological motivation*

We quickly recall how boundary link Seifert matrices appear in link theory. An m-link $L = L_1 \cup \cdots \cup L_m \subset S^{4q+3}$ is a smooth embedding of m disjoint oriented $(4q+1)$-spheres. A boundary link is a link which has m disjoint Seifert manifolds, i.e. there exist m disjoint oriented $(4q+2)$-submanifolds $F_1, \ldots, F_m \subset S^{4q+3}$ such that $\partial(F_i) = L_i, i = 1, \ldots, m$. One of the main tools for studying boundary links is the Seifert form

$$\begin{aligned} H_{2q+1}(F) \times H_{2q+1}(F) &\to \mathbb{Z} \\ (a, b) &\mapsto \mathrm{lk}(a, b_+) \end{aligned}$$

where b_+ means that we push a representative of b into $S^{4q+3} \setminus F$ along the positive normal direction of F. More precisely, we can find an orientation preserving embedding $\iota : F \times [-1, 1] \to S^{4q+3}$ and we define $a_+ = \iota(a, +1)$ and $a_- = \iota(a, -1)$.

Now pick bases $l_{i,1}, \ldots, l_{i,n_i}$ for $H_{2q+1}(F_i), i = 1, \ldots, m$, then $l_{1,1}, \ldots, l_{1,n_1}, \ldots, l_{m,1}, \ldots, l_{m,n_m}$ form a basis for $H_{2q+1}(F) = H_{2q+1}(F_1) \oplus \cdots \oplus H_{2q+1}(F_m)$. Representing the Seifert form with respect to this basis we get a boundary link Seifert matrix (cf. [14], [10, p. 670]).

We also need the notion of an F_m-link, this is a link with a map $\pi_1(S^{4q+3} \setminus L) \to F_m$, where F_m denotes the free group on m generators, which sends meridians to conjugates of the generators of F_m. A Thom argument shows that there is a one-to-one correspondence between isotopy classes of F_m-links and isotopy classes of boundary links with Seifert manifolds. It turns out that it is easier to study F_m-links, for example the addition of F_m-links is well-defined if $q > 1$. Boundary links and F_m-links are the best understood links, they have been studied thoroughly and many of the classifying results for higher dimensional knots can be done similarly in the context of such links (cf. [3, 10, 14]).

If L is a boundary link with m components then denote by $\tilde{X}$ the universal abelian cover of $S^{4q+3} \setminus L$, i.e. the cover induced by $\pi_1(S^{4q+3} \setminus L) \to H_1(S^{4q+3} \setminus L) = \mathbb{Z}^m$. Note that $H_*(\tilde{X})$ has a natural $\mathbb{Z}[\mathbb{Z}^m] = \Lambda_m$-module structure.

Proposition 1.1. *Let $L \subset S^{4q+3}$ be a boundary link with m components, and A a Seifert matrix of size $(n_1, \ldots, n_m)$ for a Seifert manifold $F = F_1 \cup \cdots \cup F_m$. Then there exists a short exact sequence*

$$0 \to \Lambda_m^n / (AT - A^t)\Lambda_m^n \to H_{2q+1}(\tilde{X}) \to P \to 0$$

where $n = \sum_{i=1}^m n_i$, P is some torsion free Λ_m-module. Furthermore $P = 0$ if $q > 0$.

We will give a quick outline of the proof which follows well-known arguments in the knot case (cf. [13, 16]).

Proof. Let $Y = S^{4q+3} \setminus F$. We can view $\tilde{X}$ as the result of gluing $\mathbb{Z}^m$ copies of Y together along $\mathbb{Z}^m$ copies of $F_1, \ldots, F_m$. Consider the resulting Mayer–Vietoris sequence

$$\cdots \to H_i(F) \otimes \Lambda_m \to H_i(Y) \otimes \Lambda_m \to H_i(\tilde{X}) \to \cdots$$
$$a_j \otimes p \mapsto (a_{j,+}t_j - a_{j,-}) \otimes p$$

where $a_j \in H_i(F_j)$. Note that for $i \in \{1, \ldots, 4q+1\}$ we have $H_i(Y) \cong H^{4q+2-i}(F) \cong H_i(F, L) \cong H_i(F)$ by Alexander duality, Poincaré duality and a long exact sequence argument. Pick a basis for $H_i(F)$ which gives A as a Seifert matrix for L, then give $H_i(Y)$ the corresponding basis. An argument as in Rolfsen (cf. [16]) shows that the map $H_{2q+1}(F) \otimes \Lambda_m \to H_{2q+1}(Y) \otimes \Lambda_m$ is given by $v \mapsto (AT - A^t)v$.

If $q = 0$ then the sequence becomes

$$\cdots \to H_1(F) \otimes \Lambda_m \to H_1(F) \otimes \Lambda_m \to H_1(\tilde{X}) \to H_0(F) \otimes \Lambda_m \to H_0(Y) \otimes \Lambda_m \to .$$

It is clear that $\operatorname{Ker}\{H_0(F) \otimes \Lambda_m \to H_0(Y) \otimes \Lambda_m\}$ is Λ_m-torsion free since $H_0(Y)$ is Λ_m-torsion free (cf. also [18]).

Now consider the case $q > 0$. We are done once we show that $H_{2q}(F) \otimes \Lambda_m \to H_{2q}(Y) \otimes \Lambda_m \cong H_{2q}(F) \otimes \Lambda_m$ is injective. Picking a basis for $H_{2q}(F)$ and giving $H_{2q}(Y)$ the corresponding basis, then we can represent this map by a matrix $B(t_1, \ldots, t_m)$. We will proof that $B(1, \ldots, 1)$ is in fact the identity matrix, in particular $\det(B) \neq 0$. This concludes the proof of the proposition. Note that $B(1, \ldots, 1)$ represents the map $H_{2q}(F) \to H_{2q}(Y)$ given by $a \mapsto a_+ - a_-$. Recall that the isomorphism $f : H_i(Y) \to H^{2q+2-i}(F)$ is induced by the linking pairing, in particular for $\sigma \in C_{2q+2-i}(F)$

$$f(a_+ - a_-)(\sigma) = \operatorname{lk}(a_+ - a_-, \sigma) = (a \times [-1, 1]) \cdot \sigma = a \cdot \sigma.$$

Thus under the Poincaré duality map $f(a_+ - a_-)$ gets sent to a. □

From the theory of fitting ideals for presentation matrices (cf. [18]), it follows that $\det(AT - A^t)$ is a well-defined invariant for a boundary link L up to multiplication by a unit in Λ_m. It is easy to see that $\det(T)^{-\frac{1}{2}} \det(AT - A^t)$ is a well-defined invariant for boundary links, it is called the Alexander polynomial of L.

Gutierrez [7, p. 34] showed that any polynomial $\Delta(t_1, \ldots, t_m)$ with the properties

$$\Delta(1, \ldots, 1) = 1$$
$$\Delta(t_1, \ldots, t_m) = \Delta\left(t_1^{-1}, \ldots, t_m^{-1}\right)$$

is the Alexander polynomial of a boundary link in dimension 1, in particular there exists a boundary link Seifert matrix A with $\Delta(A) = \Delta$. But it is difficult to find an explicit boundary link Seifert matrix, which would be important to compute further invariants.

Remark. Farber [5] and Garoufalidis and Levine [6] defined non-commutative invariants for boundary links which can be viewed as generalizations of the Alexander polynomial of a knot. Farber also proves a realization theorem.

1.3. *S-equivalence class of Seifert matrices*

In the following we will call a matrix P block diagonal, if it commutes with T, equivalently if $P = P_1 \oplus \cdots \oplus P_m$ where P_i is a $(n_i \times n_i)$-matrix.

The S-equivalence of Seifert matrices is the equivalence relation generated by the following two equivalences (for more details cf. [9, 14]).

(1) $A \sim PAP^t$ where P is a block diagonal matrix over $\mathbb{Z}$ with $\det(P) = 1$.
(2) A is equivalent to any row or column enlargement or reduction of A.

Proposition 1.2 [9, 14]. *Any two Seifert matrices for an F_m-link are S-equivalent. Furthermore any Seifert matrix is the Seifert matrix of an F_m-link.*

There exists a similar but more complicated proposition for boundary links (cf. [9]). It turns out that Seifert matrices for boundary links are related by S-equivalence and an action by (cf. also [10])

$$A_m := \{\varphi : F_m \to F_m \mid \varphi(x_i) = l_i x_i l_i^{-1} \text{ for some } l_i \in F_m\}/\text{inner automorphism}.$$

The groups A_1, A_2 are trivial [8], it follows that boundary link matrices with 2 components which are related by S-equivalence and an action by A_m are in fact S-equivalent.

It is easy to see that if A_1, A_2 are S-equivalent, then $\Delta(A_1) = \Delta(A_2)$, this shows again that the Alexander polynomial is an invariant for any F_m-link.

We call a Seifert matrix irreducible if no row or column reductions are possible.

Proposition 1.3. (1) *A Seifert matrix of size $(n_1, \ldots, n_m)$ is irreducible if and only if*

$$rank(A_{i1} \cdots A_{im}) = n_i, \quad rank \begin{pmatrix} A_{1i} \\ \vdots \\ A_{mi} \end{pmatrix} = n_i$$

for all $i = 1, \ldots, m$. Put differently, a Seifert matrix is irreducible if and only if the block columns and block rows have maximal rank.

(2) *If A_1, A_2 are S-equivalent minimal Seifert matrices then $A_1 = PA_2P^t$ where P is a block diagonal matrix over $\mathbb{Q}$ with* $\det(P) \neq 0$.

We will use this proposition to show that certain Seifert matrices are not S-equivalent.

The statement of the proposition is well-known in the case $m = 1$ (cf. [20]). The first part of the proposition is fairly straight forward to show, whereas the second part is more difficult to prove. Using ideas of Farber [4] one can rewrite the proof of Trotter in the general case, but this requires many details, which we will omit here.

2. Statement of Results

2.1. *Algebra*

For $v_1, \dots, v_l \in \mathbb{Z}$ and $\epsilon_2, \dots, \epsilon_l \in \{-1, +1\}$ define matrices $B_i := B_i(v_1, \dots, v_i, \epsilon_2, \dots, \epsilon_i)$ inductively as follows.

$$B_1 := \begin{pmatrix} v_1 & 0 \\ -1 & 1 \end{pmatrix} \quad B_i := \left(\begin{array}{ccccccccc} &&&&&&& v_i & 0 \\ &&&&&&& 0 & 1 \\ &&&&&&& v_i & 1 \\ &&&& B_{i-1} &&& 0 & 1 \\ &&&&&&& \vdots & \\ &&&&&&& v_i & 1 \\ &&&&&&& 0 & 1 \\ v_i & 0 & v_i & 0 & \dots & v_i & 0 & v_i & z_n \\ 0 & 1 & 1 & 1 & & 1 & 1 & 1-z_n & 1 \end{array}\right)$$

where $z_i := \frac{1}{2}(1 + \epsilon_i)$. Furthermore let

$$Y_l := \operatorname{diag}(y_1, y_1, y_2, y_2, \dots, y_l, y_l).$$

Proposition 2.1. *Set* $v_{l+1} = 0$, *then*

$$\det(Y_l B_l - Y_l^{-1} B_l^t) = 1 - 2v_1 + \sum_{j=1}^{l} (v_j - v_{j+1}) \left(y_1^2 \prod_{i=2}^{j} y_i^{2\epsilon_i} + y_1^{-2} \prod_{i=2}^{j} y_i^{-2\epsilon_i} \right).$$

The proof will be given in Sec. 3.

2.2. *Explanation of the algorithm*

Let $\Delta \in \mathbb{Z}[t_1, \dots, t_m]$ be a polynomial with the following properties

$$\Delta(1, \dots, 1) = 1,$$
$$\Delta(t_1, \dots, t_m) = \Delta(t_1^{-1}, \dots, t_m^{-1}).$$

Then using the usual multiindex notation we can uniquely write

$$\Delta(t_1, \dots, t_m) = \sum_{\alpha \in \mathbb{Z}^m} c_\alpha (t^\alpha + t^{-\alpha}) + 1 - \sum_{\alpha \in \mathbb{Z}^m} 2c_\alpha, \quad c_\alpha \in \mathbb{Z}$$

where $c_\alpha = 0$ for all but finitely many α and $c_{(0,\dots,0)} = 0$.

Denote the α with $c_\alpha \neq 0$ by $\alpha_1, \dots, \alpha_r$. Pick a map $p : \{0, \dots, l\} \to \mathbb{Z}^m$ with the following properties.

(1) $p(0) = (0, \dots, 0)$,
(2) $|p(t) - p(t-1)| = 1$ for all $t = 1, \dots, l$,
(3) for each $i = 1, \dots, r$ there exists a $t_i \in \{1, \dots, l\}$ such that $p(t_i) = \alpha_i$.

It is easy to see that such a map always exists. Denote the ith unit vector in $\mathbb{Z}^m$ by e_i, the second condition says that $p(t) = p(t-1) + \epsilon_t e_{s_t}$ for unique $\epsilon_t \in \{-1,+1\}, s_t \in \{1,\dots,m\}$.

Now define $w_{t_i} = c_{p(t_i)} = c_{\alpha_i}$ for $i = 1,\dots,r$ and $w_j = 0$ otherwise. Let $v_i := \sum_{j=i}^{l} w_j, j = 1,\dots,l$. From Proposition 2.1 it follows now immediately that for $B = B(v_1,\dots,v_l,\epsilon_2,\dots,\epsilon_l)$ and $Y := \operatorname{diag}(y_{s_1},\dots,y_{s_l})$, we get

$$\det(YB - Y^{-1}B^t) = 1 - 2v_1 + \sum_{j=1}^{l}(v_j - v_{j+1})\left(y_{s_1}^2 \prod_{i=2}^{j} y_{s_i}^{2\epsilon_i} + y_{s_1}^{-2} \prod_{i=2}^{j} y_{s_i}^{-2\epsilon_i}\right).$$

Using multiindex notation $y = (y_1,\dots,y_m)$, we can rewrite this as

$$\sum_{j=1}^{l} w_j(y^{p(j)} + y^{-p(j)}) + 1 - \sum_{j=1}^{l} 2w_j = \sum_{\alpha\in\mathbb{Z}^m} c_\alpha(y^{2\alpha} + y^{-2\alpha}) + 1 - \sum_{\alpha\in\mathbb{Z}^m} 2c_\alpha$$

in particular for $\tilde{T} := \operatorname{diag}(t_{s_1},\dots,t_{s_l})$, we get

$$\det(\tilde{T})^{\frac{1}{2}} \det(\tilde{T}B - B^t) = \sum_{\alpha\in\mathbb{Z}^m} c_\alpha(t^\alpha + t^{-\alpha}) + 1 - \sum_{\alpha\in\mathbb{Z}^m} 2c_\alpha = \Delta.$$

We can find a permutation matrix P such that

$$P\tilde{T}P^{-1} = \operatorname{diag}\left(\underbrace{t_1,\dots,t_1}_{n_1},\dots,\underbrace{t_m,\dots,t_m}_{n_m}\right) =: T$$

for some $n_1,\dots,n_m$. In fact we can and will assume that P is of form

$$\begin{aligned} &P(v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2},\dots,v_{l,1},v_{l,2}) \\ &\quad = P(v_{\sigma(1),1}, v_{\sigma(1),2}, v_{\sigma(2),1}, v_{\sigma(2),2},\dots,v_{\sigma(l),1},v_{\sigma(l),2}) \end{aligned}$$

for some permutation $\sigma \in S_l$, i.e. P permutes pairs of coordinates. Note that $P^t = P^{-1}$ and $\det(P) = 1$.

Theorem 2.2. *The matrix $A = PBP^{-1}$ is a boundary link Seifert matrix of size $(n_1,\dots,n_m)$ and $\Delta(A) = \Delta$.*

Proof. Note that $B - B^t$ and hence $A - A^t$ is a block sum of 2×2 matrices of the form $\left(\begin{smallmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{smallmatrix}\right)$, in particular A is a Seifert matrix of size $(n_1,\dots,n_m)$, furthermore

$$\begin{aligned} \Delta(A) &= \det(T)^{-\frac{1}{2}} \det(TA - A^t) = \det(T)^{-\frac{1}{2}} \det(PTP^{-1}PAP^{-1} - PA^tP^{-1}) \\ &= \det(\tilde{T})^{-\frac{1}{2}} \det(\tilde{T}B - B^t) = \Delta. \end{aligned}$$

□

Using Proposition 1.2 we get the following corollary.

Corollary 2.3. *Any polynomial Δ with $\Delta(1,\dots,1) = 1$ and $\Delta(t_1^{-1},\dots,t_m^{-1}) = \Delta(t_1,\dots,t_m)$ is the Alexander polynomial of a boundary link.*

It is clear that A depends on the map p, for example A is a $(2l \times 2l)$-smatrix, i.e. p determines the size of A. We will see in the next section that different paths can in fact give non S-equivalent matrices.

2.3. *Example*

2.3.1. *Minimality of matrices*

Let $\Delta = c_{1,0}\big(t_1+t_1^{-1}\big)+c_{1,1}\big(t_1t_2+t_1^{-1}t_2^{-1}\big)+c_{0,1}\big(t_2^2+t_2^{-2}\big)-17$, then $\alpha_1 = (1,0), \alpha_2 = (1,1), \alpha_3 = (0,1)$. The map $p(0) := (0,0), p(1) := (1,0), p(2) := (1,1), p(3) := (0,1)$ satisfies the conditions on p. In this case

$$\begin{aligned}
&t_1 = 1, && t_2 = 2, && t_3 = 3,\\
&s_1 = 1, && s_2 = 2, && s_3 = 1,\\
&\epsilon_1 = 1, && \epsilon_2 = 1, && \epsilon_3 = -1,\\
&w_1 = c_{1,0}, && w_2 = c_{1,1}, && w_3 = c_{0,1},\\
&v_1 = c_{1,0} + c_{1,1} + c_{0,1}, && v_2 = c_{1,1} + c_{0,1}, && v_3 = c_{0,1}.
\end{aligned}$$

Then

$$B = \begin{pmatrix} v_1 & 0 & v_2 & 0 & v_3 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ v_2 & 0 & v_2 & 1 & v_3 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 0 & v_3 & 0 & v_3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} v_1 & 0 & v_3 & 0 & v_2 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 0 & v_3 & 0 & v_3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ v_2 & 0 & v_3 & 1 & v_2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

where we chose $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$.

Using Proposition 1.3, it is easy to see that A forms an irreducible Seifert matrix of size $(2, 1)$.

Consider

$$A = \begin{pmatrix} w_1 + w_3 & 0 & -w_3 & 0 \\ -1 & 1 & 0 & 1 \\ -w_3 & 0 & w_2 + w_3 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

then

$$\Delta(A) = w_1(t_1 + t_1^{-1}) + w_2(t_2 + t_2^{-1}) + w_3(t_1t_2 + t_1^{-1}t_2^{-1}) + 1 - 2(w_1 + w_2 + w_3).$$

This shows that the algorithm does in general not produce a Seifert matrix of minimal size for a given Alexander polynomial.

2.3.2. *Uniqueness of result*

A straight forward argument shows that for a knot Alexander polynomial $\Delta(t)$ different choices of maps p will produce S-equivalent matrices. This is no longer true in the case $m > 1$.

Consider $\Delta = w(t_1t_2 + t_1^{-1}t_2^{-1}) + 1 - 2w, w \neq 0$. If we take maps p_1, p_2 with $p_1(0) = (0,0), p_1(1) = (1,0)$ and $p_1(2) = (1,1)$ and $p_2(0) = (0,0), p_2(1) = (0,1)$ and $p_2(2) = (1,1)$ then applying the algorithm we will get identical matrices B but we have to use different permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

We get Seifert matrices

$$A_1 = \begin{pmatrix} w & 1 & w & 0 \\ 0 & 1 & 1 & 1 \\ w & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 & w & 1 \\ 0 & 1 & 0 & 1 \\ w & 0 & w & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Both matrices are minimal, but not block congruent, since $\det(A_{1,11}) = w$, $\det(A_{2,11}) = 0$. Hence by Proposition 1.3, A_1 and A_2 are not S-equivalent.

Recall that any boundary link Seifert matrix corresponds to an F_m-link, we therefore can construct non-isotopic F_m-links with identical Alexander polynomials. I do not know whether the matrices are S_m-equivalent, in particular whether the corresponding boundary links are isotopic.

Using signature invariants one can show that these matrices are in fact not even matrix cobordant (for a definition, cf. [10]), i.e. one can show that the corresponding F_m-links are in fact not even F_m-cobordant.

3. Proof of Proposition 2.1

3.1. *Proof of a special case of Proposition 2.1*

In this section we will consider the case $\epsilon_2 = \cdots = \epsilon_l = 1$, we have to show that

$$\det\big(Y_lB_l - Y_l^{-1}B_l^t\big) = 1 - 2v_1 + \sum_{j=1}^{l}(v_j - v_{j+1})\left(\prod_{i=1}^{j} y_i^2 + \prod_{i=1}^{j} y_i^{-2}\right).$$

We will show how to compute the determinant, but we will give the matrices only for the case $l = 4$ to simplify the notation.

Consider $Y_lB_l - Y_l^{-1}B_l^t$:

$$\begin{pmatrix}
v_1(y_1 - y_1^{-1}) & y_1^{-1} & v_2(y_1 - y_1^{-1}) & 0 & v_3(y_1 - y_1^{-1}) & 0 & v_4(y_1 - y_1^{-1}) & 0 \\
-y_1 & y_1 - y_1^{-1} & 0 & y_1 - y_1^{-1} & 0 & y_1 - y_1^{-1} & 0 & y_1 - y_1^{-1} \\
v_2(y_2 - y_2^{-1}) & 0 & v_2(y_2 - y_2^{-1}) & y_2 & v_3(y_2 - y_2^{-1}) & y_2 - y_2^{-1} & v_4(y_2 - y_2^{-1}) & y_2 - y_2^{-1} \\
0 & y_2 - y_2^{-1} & -y_2^{-1} & y_2 - y_2^{-1} & 0 & y_2 - y_2^{-1} & 0 & y_2 - y_2^{-1} \\
v_3(y_3 - y_3^{-1}) & 0 & v_3(y_3 - y_3^{-1}) & 0 & v_3(y_3 - y_3^{-1}) & y_3 & v_4(y_3 - y_3^{-1}) & y_3 - y_3^{-1} \\
0 & y_3 - y_3^{-1} & y_3 - y_3^{-1} & y_3 - y_3^{-1} & -y_3^{-1} & y_3 - y_3^{-1} & 0 & y_3 - y_3^{-1} \\
v_4(y_4 - y_4^{-1}) & 0 & v_4(y_4 - y_4^{-1}) & 0 & v_4(y_4 - y_4^{-1}) & 0 & v_4(y_4 - y_4^{-1}) & y_4 \\
0 & y_4 - y_4^{-1} & y_4 - y_4^{-1} & y_4 - y_4^{-1} & y_4 - y_4^{-1} & y_4 - y_4^{-1} & -y_4^{-1} & y_4 - y_4^{-1}
\end{pmatrix}.$$

We will first simplify the matrix to make the computation of the determinant easier. For $i = 2, \dots, l$ multiply the second row by $\frac{y_i - y_i^{-1}}{y_1 - y_1^{-1}}$ and subtract the result from the $2i$th row, we get

$$\begin{pmatrix}
v_1(y_1 - y_1^{-1}) & y_1^{-1} & v_2(y_1 - y_1^{-1}) & 0 & v_3(y_1 - y_1^{-1}) & 0 & v_4(y_1 - y_1^{-1}) & 0 \\
-y_1 & y_1 - y_1^{-1} & 0 & y_1 - y_1^{-1} & 0 & y_1 - y_1^{-1} & 0 & y_1 - y_1^{-1} \\
v_2(y_2 - y_2^{-1}) & 0 & v_2(y_2 - y_2^{-1}) & y_2 & v_3(y_2 - y_2^{-1}) & y_2 - y_2^{-1} & v_4(y_2 - y_2^{-1}) & y_2 - y_2^{-1} \\
y_1 \frac{y_2 - y_2^{-1}}{y_1 - y_1^{-1}} & 0 & -y_2^{-1} & 0 & 0 & 0 & 0 & 0 \\
v_3(y_3 - y_3^{-1}) & 0 & v_3(y_3 - y_3^{-1}) & 0 & v_3(y_3 - y_3^{-1}) & y_3 & v_4(y_3 - y_3^{-1}) & y_3 - y_3^{-1} \\
y_1 \frac{y_3 - y_3^{-1}}{y_1 - y_1^{-1}} & 0 & y_3 - y_3^{-1} & 0 & -y_3^{-1} & 0 & 0 & 0 \\
v_4(y_4 - y_4^{-1}) & 0 & v_4(y_4 - y_4^{-1}) & 0 & v_4(y_4 - y_4^{-1}) & 0 & v_4(y_4 - y_4^{-1}) & y_4 \\
y_1 \frac{y_4 - y_4^{-1}}{y_1 - y_1^{-1}} & 0 & y_4 - y_4^{-1} & 0 & y_4 - y_4^{-1} & 0 & -y_4^{-1} & 0
\end{pmatrix}.$$

For $i = 1, \dots, l-1$ subtract the $(2i+1)$st column from the $(2i-1)$st column and for $i = l-1, \dots, 1$ subtract the $2i$th column from the $(2i+2)$nd column, we get

$$\begin{pmatrix}
w_1(y_1 - y_1^{-1}) & y_1^{-1} & w_2(y_1 - y_1^{-1}) & -y_1^{-1} & w_3(y_1 - y_1^{-1}) & 0 & w_4(y_1 - y_1^{-1}) & 0 \\
-y_1 & y_1 - y_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & w_2(y_2 - y_2^{-1}) & y_2 & w_3(y_2 - y_2^{-1}) & -y_2^{-1} & w_4(y_2 - y_2^{-1}) & 0 \\
y_1 \frac{y_2 - y_2^{-1}}{y_1 - y_1^{-1}} + y_2^{-1} & 0 & -y_2^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & w_3(y_3 - y_3^{-1}) & y_3 & w_4(y_3 - y_3^{-1}) & -y_3^{-1} \\
y_1 \frac{y_3 - y_3^{-1}}{y_1 - y_1^{-1}} - (y_3 - y_3^{-1}) & 0 & y_3 & 0 & -y_3^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & w_4(y_4 - y_4^{-1}) & y_4 \\
y_1 \frac{y_4 - y_4^{-1}}{y_1 - y_1^{-1}} - (y_4 - y_4^{-1}) & 0 & 0 & 0 & y_4 & 0 & -y_4^{-1} & 0
\end{pmatrix}$$

where $w_i := v_i - v_{i+1}, i = 1, \dots, l-1$, recall that $v_{l+1} = 0$, hence $w_l := v_l$. For $i = 2, \dots, l$ multiply the $(2i-1)$st row by $\frac{y_{i-1} - y_{i-1}^{-1}}{y_i - y_i^{-1}}$ and subtract the result from the $(2i-3)$rd row, furthermore for $i = 2, \dots, l-1$ multiply the $2i$th row by $y_i y_{i+1}$ and subtract the result from the $(2i+2)$nd row. An induction argument shows that the result is a matrix D_l which is inductively defined as follows.

$$D_1 = \begin{pmatrix} w_1(y_1 - y_1^{-1}) & y_1^{-1} \\ -y_1 & y_1 - y_1^{-1} \end{pmatrix},$$

$$D_2 = \left(\begin{array}{cccc}
\multicolumn{2}{c}{D_1} & 0 & \frac{y_1^{-1} y_2^{-1} - y_1 y_2}{y_2 - y_2^{-1}} \\
 & & 0 & 0 \\
0 & 0 & w_2(y_2 - y_2^{-1}) & y_2 \\
\frac{-y_1^{-1} y_2^{-1} + y_1 y_2}{y_1 - y_1^{-1}} & 0 & -y_2^{-1} & 0
\end{array}\right),$$

and for $n = 3, \dots, l$

$$D_n = \begin{pmatrix} & & & & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & D_{n-1} & & & 0 & \frac{y_{n-1}^{-1}(y_{n-2}-y_{n-2}^{-1})}{y_{n-1}-y_{n-1}^{-1}} \\ & & & & & 0 & 0 \\ & & & & & 0 & \frac{y_{n-1}^{-1}y_n^{-1}-y_{n-1}y_n}{y_n-y_n^{-1}} \\ & & & & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & w_n(y_n - y_n^{-1}) & y_n \\ \frac{-y_1^{-1}y_n^{-1}+y_1y_2^2\cdot\dots\cdot y_{n-1}^2y_n}{y_1-y_1^{-1}} & 0 & 0 & \cdots & 0 & -y_n^{-1} & 0 \end{pmatrix}.$$

Note that $\det(D_l) = \det(Y_lB_l - Y_l^{-1}B_l^t)$, we will now compute $\det(D_l)$. For $n = 2, \dots, l$ we denote by D'_n (respectively, D''_n) the matrix obtained from D_n by deleting the first column and the $(2n-3)$rd (respectively, $(2n-1)$st) row. Define

$$\det_n := \det(D_n), \quad \det'_n := \det(D'_n), \quad \det''_n := \det(D''_n).$$

Using the last row to compute $\det(D_n)$ we get

$$\begin{aligned} \det_n = \det_{n-1} - w_n(y_n - y_n^{-1})\frac{-y_1^{-1}y_n^{-1} + y_1y_2^2\cdot\dots\cdot y_{n-1}^2y_n}{y_1 - y_1^{-1}} \\ \times\left(\frac{y_{n-1}^{-1}(y_{n-2} - y_{n-2}^{-1})}{y_{n-1} - y_{n-1}^{-1}}\det'_{n-1} + \frac{y_{n-1}^{-1}y_n^{-1} - y_{n-1}y_n}{y_n - y_n^{-1}}\det''_{n-1}\right). \end{aligned}$$

We make the following easy observations:

$$\begin{aligned} \det'_n &= \det''_{n-1}, \\ \det''_n &= -y_n^{-1}\left(\frac{y_{n-1}^{-1}(y_{n-2} - y_{n-2}^{-1})}{y_{n-1} - y_{n-1}^{-1}}\det'_{n-1} + \frac{y_{n-1}^{-1}y_n^{-1} - y_{n-1}y_n}{y_n - y_n^{-1}}\det''_{n-1}\right). \end{aligned}$$

It follows that

$$\det_n = \det_{n-1} - w_n(y_n - y_n^{-1})\frac{-y_1^{-1}y_n^{-1} + y_1y_2^2\cdot\dots\cdot y_{n-1}^2y_n}{y_1 - y_1^{-1}}y_n\det''_n.$$

Recall that we have to show that

$$\begin{aligned} \det_n &= 1 - 2v_1 + \sum_{j=1}^{l}(v_j - v_{j+1})\left(\prod_{i=1}^{j}y_i^2 + \prod_{i=1}^{j}y_i^{-2}\right) \\ &= 1 - 2\sum_{j=1}^{l}w_j + \sum_{j=1}^{l}w_j\left(\prod_{i=1}^{j}y_i^2 + \prod_{i=1}^{j}y_i^{-2}\right). \end{aligned}$$

The proof of the special case of Proposition 2.1 is complete once we show that

$$\begin{aligned}
&\det{}_1 = w_1(y_1^2 + y_1^{-2}) + 1 - 2w_1 \\
&\frac{-y_1^{-1}y_n^{-1} + y_1y_2^2 \cdot \cdots \cdot y_{n-1}^2 y_n}{y_1 - y_1^{-1}} y_n(y_n - y_n^{-1}) \det{}''_n \\
&\quad = y_1^2 \cdot \cdots \cdot y_n^2 + y_1^{-2} \cdot \cdots \cdot y_n^{-2} - 2 \qquad \text{for } n = 2, \ldots, l.
\end{aligned}$$

The first equality follows from a simple computation. We now prove the second equality by induction on n. For $n = 1, 2$ this follows again from a direct computation. Now assume that the statement is true for all $k < n$, then using the above results we get

$$\begin{aligned}
&\frac{-y_1^{-1}y_n^{-1} + y_1y_2^2 \cdot \cdots \cdot y_{n-1}^{-1}}{y_1 - y_1^{-1}} y_n(y_n - y_n^{-1}) \det{}''_n \\
&= \frac{-y_1^{-1}y_n^{-1} + y_1y_2^2 \cdot \cdots \cdot y_{n-1}^{-1}}{y_1 - y_1^{-1}} (y_n - y_n^{-1}) \det{}''_n \\
&\quad \times \left(\frac{-y_{n-1}(y_{n-2} - y_{n-2}^{-1})}{y_{n-1} - y_{n-1}^{-1}} \det{}''_{n-2} - \frac{y_{n-1}^{-1} - y_{n-1}y_n}{y_n - y_n^{-1}} \det{}''_{n-1} \right).
\end{aligned}$$

Using the induction hypothesis, we get an expression in the five variables $y_1, y_2^2 \cdot \cdots \cdot y_{n-3}^2, y_{n-2}, y_{n-1}, y_n$ which can be computed to equal $y_1^2 \cdot \cdots \cdot y_n^2 + y_1^{-2} \cdot \cdots \cdot y_n^{-2} - 2$.

3.2. *Proof of Proposition 2.1*

Let $\epsilon_2, \ldots, \epsilon_l \in \{-1, +1\}$. Denote by $\varphi : \mathbb{Z}[y_1^{\pm 1}, \ldots, y_l^{\pm 1}] \to \mathbb{Z}[y_1^{\pm 1}, \ldots, y_l^{\pm 1}]$ the ring homomorphism induced by $\varphi(y_1) = y_1$ and $\varphi(y_i) = y_i^{\epsilon_i}, i = 2, \ldots, l$, denote the induced map on $M_{2m \times 2m}(\mathbb{Z}[y_1^{\pm 1}, \ldots, y_l^{\pm 1}])$ by φ as well. Write $B(\epsilon_2, \ldots, \epsilon_l)$ for $B(v_1, \ldots, v_l, \epsilon_2, \ldots, \epsilon_l)$.

We see that if we multiply the $(2i-1)$st and the $2i$th row of $Y_l B(\epsilon_2, \ldots, \epsilon_l) - Y_l^{-1} B(\epsilon_2, \ldots, \epsilon_l)^t$ by $\epsilon_i, i = 2, \ldots, l$, then we get $\varphi(Y_l B(1, \ldots, 1) - Y_l^{-1} B(1, \ldots, 1)^t)$, in particular the determinants are the same, i.e.

$$\begin{aligned}
&\det\bigl(Y_l B(\epsilon_2, \ldots, \epsilon_l) - Y_l^{-1} B(\epsilon_2, \ldots, \epsilon_l)^t\bigr) \\
&\quad = \varphi\bigl(\det(Y_l B(1, \ldots, 1) - Y_l^{-1} B(1, \ldots, 1)^t)\bigr) \\
&\quad = \varphi\left(1 - 2v_1 + \sum_{j=1}^{l} (v_j - v_{j+1}) \left(\prod_{i=1}^{j} y_i^2 + \prod_{i=1}^{j} y_i^{-2} \right)\right) \\
&\quad = 1 - 2v_1 + \sum_{j=1}^{l} (v_j - v_{j+1}) \left(y_1^2 \prod_{i=2}^{j} y_i^{2\epsilon_i} + y_1^{-2} \prod_{i=2}^{j} y_i^{-2\epsilon_i} \right).
\end{aligned}$$

This proves Proposition 2.1.

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